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REGULARITY OF THE INVERSE OF A SOBOLEV HOMEOMORPHISM

ROSSELLA CORPORENTE

We establish a connection between A_Φ and G_Ψ classes of weights (see [13], [23]) in the context of Orlicz spaces. We give a generalization of the following result (see [11])

$$h' \in A_p \Longleftrightarrow (h^{-1})' \in G_{p'}, \ \ \frac{1}{p}+\frac{1}{p'}=1$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism such that $h, h^{-1} \in W^{1,1}_{loc}(\mathbb{R})$, A_p and $G_{p'}$ are, respectively, Muckenhoupt and Gehring classes of weights (see [18], [9]).

1. Introduction

In this paper we study some weighted integral inequalities in Orlicz Spaces. More specifically we extend to the context of Orlicz classes the following result of Johnson and Neugebauer (see [11]):

$$h' \in A_p \Longleftrightarrow (h^{-1})' \in G_{p'}, \ \ \frac{1}{p}+\frac{1}{p'}=1 \tag{1}$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism such that $h, h^{-1} \in W^{1,1}_{loc}(\mathbb{R})$. Further they proved the equality between the corresponding constants:

$$A_p(h') = G_{p'}(h^{-1})'.$$

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The aim of this paper is to prove that condition (1) is also true for more general Young functions and not only for power functions. If Φ e Ψ are complementary Young functions, the following definitions hold

$$h' \in A_\Phi \Leftrightarrow \exists A \geq 1 : \forall \varepsilon > 0, \left(\fint_I \varepsilon h' ds\right) \varphi\left(\fint_I \varphi^{-1}\left(\frac{1}{\varepsilon h'}\right) ds\right) \leq A,\ \varphi(t) = \Phi'(t)$$

and

$$(h^{-1})' \in G_\Psi \Longleftrightarrow \exists B \geq 1 : \forall \varepsilon > 0, \frac{\Psi^{-1}\left(\fint_J \Psi\left(\frac{(h^{-1})'}{\varepsilon}\right) ds\right)}{\left(\fint_J \frac{(h^{-1})'}{\varepsilon} ds\right)} \leq B,$$

$\forall\, I, J$ bounded intervals of $\mathbb{R}$, where $\fint_I = \frac{1}{|I|}\int_I$ and $|E|$ denotes the classical Lebesgue measure.

At first we prove that, if the growth of Φ is close to a power function from the point of view of indices (see (36)),

$$(h^{-1})' \in G_\Psi \Longrightarrow h' \in A_\Phi,$$

where $h : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism such that $h, h^{-1} \in W^{1,1}_{loc}(\mathbb{R})$ and Φ, Ψ are complementary Young functions verifying Δ_2-condition.

Then we prove the converse inequality for a certain class of Young functions. Namely, setting

$$\varphi_{p,\alpha}(s) = \frac{s^p}{\log^\alpha(e+s)},\ \alpha \geq 0,\ p > 1$$

$$\Phi_{p,\alpha}(t) = \int_0^t \varphi_{p,\alpha}(s)\, ds$$

$$\Psi_{p,\alpha}(t) = \text{complementary function of } \Phi_{p,\alpha}(t),$$

we get the following statement

$$\forall M > 1\ \exists\, \alpha \geq 0 : h' \in A_{\Phi_{p,\alpha}},\ A_{\Phi_{p,\alpha}}(h') \leq M \Longrightarrow (h^{-1})' \in G_{\Psi_{p,\alpha}}.$$

In conclusion, in this paper, we establish a connection between A_Φ and G_Ψ classes, giving a generalization of condition (1).

2. Preliminaries

A Young function is a convex function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ such that Φ is increasing on $[0,\infty)$, satisfying

$$\lim_{t\to 0}\frac{\Phi(t)}{t} = 0, \qquad \lim_{t\to\infty}\frac{\Phi(t)}{t} = \infty.$$

Φ has a derivative φ which is nondecreasing and nonnegative, $\varphi(0+) = 0$ and $\varphi(\infty) = \infty$, so that

$$\Phi(t) = \int_0^t \varphi(x)dx$$

and we can take φ to be right-continuous. The Young function complementary to Φ is defined by

$$\Psi(t) = \sup_{s>0}\{st - \Phi(s)\} = \int_0^t \psi(x)dx,$$

where $\psi(x) = \inf\{s : \varphi(s) \geq x\}$. These functions verify the Young's inequality

$$ab \leq \Phi(a) + \Psi(b) \qquad \forall\, a,b > 0.$$

The Young function Φ satisfies the Δ_2-condition (we will write $\Phi \in \Delta_2$) if there exists a constant $c > 0$ such that

$$\Phi(2t) \leq c\,\Phi(t), \quad \forall t \geq 0. \tag{2}$$

The following result shows that we can substitute Δ_2-condition with a growth condition that gives a control of a Young function by power functions. This is frequently used, for instance, in connections with applications to PDEs (see, e.g. [8], [21], [22]).

Proposition 2.1. *[16] Let Φ be a Young function, then*

$$\Phi \in \Delta_2 \Longleftrightarrow \exists\, p,q, 1 \leq p \leq q : p\Phi(t) \leq t\,\Phi'(t) \leq q\,\Phi(t) \quad \forall t > 0. \tag{3}$$

Note that the growth condition (3) means also that the function $\frac{\Phi(t)}{t^p}$ is nondecreasing and $\frac{\Phi(t)}{t^q}$ is nonincreasing.

Next two results deal with relations between complementary Young functions and Hölder conjugate exponents. Here and for all the sequel p' (and similarly q', r') will denote the conjugate exponent $p' = \frac{p}{p-1}$.

Proposition 2.2. *[16] Let $1 < p \leq q$ and let Φ and Ψ be complementary Young functions and suppose that their derivatives are continuous, then*

$$p\,\Psi(t) \leq t\,\Psi'(t) \leq q\,\Psi(t) \Longleftrightarrow q'\,\Phi(t) \leq t\,\Phi'(t) \leq p'\,\Phi(t), \quad \forall\, t > 0. \tag{4}$$

Proposition 2.3. *Let $1 < p \le r < \infty$ and let Φ, Ψ be complementary Young functions verifying the Δ_2-condition. If $\varphi(t) = \Phi'(t)$ verifies*

$$(r'-1)\varphi(t) \le t\varphi'(t), \quad \forall t > 0 \tag{5}$$

then

$$r'\Phi(t) \le t\varphi(t), \quad \forall t > 0 \tag{6}$$

and therefore

$$t\Psi'(t) \le r\Psi(t), \quad \forall t > 0. \tag{7}$$

Proof. Integrating (5), we get

$$(r'-1)\Phi(t) \le \int_0^t s\varphi'(s)ds, \quad \forall t > 0$$

and by integration by parts

$$(r'-1)\Phi(t) \le t\varphi(t) - \Phi(t), \quad \forall t > 0$$

i.e. (6) holds true and by (4) we get (7). □

One more result about the growth condition (3) will be used in the following.

Proposition 2.4. *[5] Let $1 \le p \le q$. If Φ is a Young function verifying the growth condition*

$$p\Phi(t) \le t\Phi'(t) \le q\Phi(t), \quad \forall\, t > 0,$$

then $\exists C > 0$ such that

$$\Phi(\lambda t) \le C\max\{\lambda^p, \lambda^q\}\Phi(t), \quad \forall\, \lambda, t > 0. \tag{8}$$

Next theorem states that, as a consequence of the growth condition, the Jensen mean $\Psi^{-1}\left(\fint_Q \Psi(w)dx\right)$ lies between the L_p- norm and the L_q-norm.

Theorem 2.5. *[5] Let $1 < p \le q$ and let $w \in L^1_{loc}(\mathbb{R})$ be nonnegative. If Ψ is a Young function verifying the condition*

$$p\Psi(t) \le t\Psi'(t) \le q\Psi(t), \quad \forall\, t > 0$$

then

$$\frac{1}{C}\left(\fint_I w^p dx\right)^{\frac{1}{p}} \le \Psi^{-1}\left(\fint_I \Psi(w)dx\right) \le C\left(\fint_I w^q dx\right)^{\frac{1}{q}} \tag{9}$$

where $C = (\frac{q}{p})^{\frac{1}{p}}$.

We conclude this Section introducing the notion of fundamental indices, which will play a key role in the sequel.

Setting

$$h_\Phi(\lambda) = \sup_{t>0} \frac{\Phi(\lambda t)}{\Phi(t)}, \quad \lambda > 0 \tag{10}$$

the numbers

$$\underline{\alpha}(\Phi) = \lim_{\lambda \to 0^+} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \sup_{0<\lambda<1} \frac{\log h_\Phi(\lambda)}{\log \lambda} \tag{11}$$

and

$$\overline{\alpha}(\Phi) = \lim_{\lambda \to \infty} \frac{\log h_\Phi(\lambda)}{\log \lambda} = \inf_{1<\lambda<\infty} \frac{\log h_\Phi(\lambda)}{\log \lambda} \tag{12}$$

are called the lower index of Φ and the upper index of Φ, respectively.

The fundamental indices $\underline{\alpha}(\Phi)$ and $\overline{\alpha}(\Phi)$ are reciprocals of the Boyd indices (see [1], [14]). In the same way we can define the fundamental indices of Ψ, $\underline{\alpha}(\Psi)$ and $\overline{\alpha}(\Psi)$. We have the following relations:

$$1 \leq \underline{\alpha}(\Phi) \leq \overline{\alpha}(\Phi) \text{ and } \underline{\alpha}(\Phi) > 1 \Longleftrightarrow \Psi \in \Delta_2 \tag{13}$$

$$1 \leq \underline{\alpha}(\Psi) \leq \overline{\alpha}(\Psi) \text{ and } \underline{\alpha}(\Psi) > 1 \Longleftrightarrow \Phi \in \Delta_2. \tag{14}$$

Moreover, the couples $\underline{\alpha}(\Psi)$, $\overline{\alpha}(\Phi)$, and $\overline{\alpha}(\Psi)$, $\underline{\alpha}(\Phi)$ behave similarly as conjugate exponents of power functions, namely:

$$\underline{\alpha}(\Psi) = \frac{\overline{\alpha}(\Phi)}{\overline{\alpha}(\Phi) - 1} \tag{15}$$

and

$$\overline{\alpha}(\Psi) = \frac{\underline{\alpha}(\Phi)}{\underline{\alpha}(\Phi) - 1}.$$

For the sake of completeness, we conclude this Section stating a couple of results which help to understand how indices are related to the growth of the Young functions.

The following theorem gives a simple formula to compute the fundamental indices of a given Young function.

Theorem 2.6. *[7] If there exist*

$$r_0 = \lim_{t \to 0} \frac{t\,\Phi'(t)}{\Phi(t)} \quad \textit{and} \quad r_\infty = \lim_{t \to \infty} \frac{t\,\Phi'(t)}{\Phi(t)}$$

then

$$\underline{\alpha}(\Phi) = \min\{r_0, r_\infty\} \quad \textit{and} \quad \overline{\alpha}(\Phi) = \max\{r_0, r_\infty\}.$$

Example 2.7. The Young function $\Phi(t) = |t|^p \log^\alpha(a+|t|)$, with $1 < p < \infty$ and $\alpha \geq 0$ has the following fundamental indices:

$$\begin{cases} \underline{\alpha}(\Phi) = \overline{\alpha}(\Phi) = p, & \text{if } a > 1, \\ \underline{\alpha}(\Phi) = p,\ \overline{\alpha}(\Phi) = p+\alpha, & \text{if } a = 1. \end{cases}$$

Example 2.8. The Young function $\Phi(t) = t^2 - \frac{\log(1+t^2)}{2}$, $t \geq 0$ has lower and upper index equal to 2.

Proposition 2.9. *[6] Let Φ be a Young function satisfying the growth condition $p\Phi(t) \leq t\Phi'(t) \leq q\Phi(t)$, $\forall t > 0$, with $1 < p \leq q$, then we have*

$$p \leq \underline{\alpha}(\Phi) \leq \overline{\alpha}(\Phi) \leq q. \tag{16}$$

where $\underline{\alpha}(\Phi)$ and $\overline{\alpha}(\Phi)$ are the fundamental indices of Φ (see (11) *and* (12)*).*

3. A_Φ and G_Ψ classes

Let us recall the definition of the A_p-class introduced by Muckenhoupt in [18]. Let $1 < p < \infty$. A weight w (i.e. a positive function in $L^1_{loc}(\mathbb{R})$) belongs to the A_p-class if there exists $A \geq 1$ such that

$$⨍_I w\,dx \left(⨍_I w^{-\frac{1}{p-1}}dx \right)^{p-1} \leq A, \tag{17}$$

for all bounded intervals I in $\mathbb{R}$. The constant $A_p(w)$ is defined by

$$A_p(w) = \sup_I ⨍_I w\,dx \left(⨍_I w^{-\frac{1}{p-1}}dx \right)^{p-1},$$

where the supremum is taken over all bounded intervals $I \subset \mathbb{R}$.

If $p = \infty$, a weight w belongs to the A_∞-class if there exists $A \geq 1$ such that

$$\left(⨍_I w\,dx \right) \left(\exp ⨍_I \log \frac{1}{w} dx \right) \leq A, \tag{18}$$

for all bounded intervals I in $\mathbb{R}$. The constant $A_\infty(w)$ is defined by

$$A_\infty(w) = \sup_I \left(⨍_I w\,dx \right) \left(\exp ⨍_I \log \frac{1}{w} dx \right), \tag{19}$$

where the supremum is taken over all bounded intervals $I \subset \mathbb{R}$.

In [13], Kerman and Torchinsky extended this definition in the framework of the Orlicz spaces. Let w be a weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition. We say that w belongs to the $A_\Phi-$class (we write $w \in A_\Phi$) if there exists $A \geq 1$ such that

$$\forall \varepsilon > 0, \left(\fint_I \varepsilon w dx \right) \varphi \left(\fint_I \varphi^{-1} \left(\frac{1}{\varepsilon w} \right) dx \right) \leq A, \tag{20}$$

for all bounded intervals I in $\mathbb{R}$, where $\Phi' = \varphi$. The constant $A_\Phi(w)$ is defined by

$$A_\Phi(w) = \sup_{\varepsilon>0} \sup_I \left[\fint_I \varepsilon w dx \, \varphi \left(\fint_I \varphi^{-1} \left(\frac{1}{\varepsilon w} \right) dx \right) \right]. \tag{21}$$

On the other hand, we have the G_q-class introduced by Gehring [9], and the related constant (see [17]). Let $1 < q < \infty$. A weight v belongs to G_q-class, if there exists a constant $B \geq 1$ such that

$$\frac{\left(\fint_I v^q dx \right)^{\frac{1}{q}}}{\fint_I v dx} \leq B, \tag{22}$$

for all bounded intervals I in $\mathbb{R}$. The related constant is defined by

$$G_q(v) = \sup_I \left[\frac{\left(\fint_I v^q dx \right)^{\frac{1}{q}}}{\fint_I v dx} \right]^{q'}, \tag{23}$$

where the supremum is taken over all bounded intervals $I \subset \mathbb{R}$.

If $q = 1$, a weight v belongs to G_1-class, if there exists a constant $B \geq 1$ such that

$$\left(\exp \fint_I \frac{v}{v_I} \log \frac{v}{v_I} dx \right) \leq B, \tag{24}$$

for all bounded intervals I in $\mathbb{R}$. The related constant is defined by

$$G_1(v) = \sup_I \left(\exp \fint_I \frac{v}{v_I} \log \frac{v}{v_I} dx \right), \tag{25}$$

with $v_I = \fint_I v \, dx$, where the supremum is taken over all bounded intervals $I \subset \mathbb{R}$.

As before it is possible to extend this definition to Orlicz spaces. Let v be a weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition.

We say that $v \in G_\Psi$ if there exists $B \geq 1$ such that

$$\forall \varepsilon > 0, \; \frac{\Psi^{-1}\left(\fint_I \Psi\left(\frac{v}{\varepsilon}\right) dx \right)}{\fint_I \frac{v}{\varepsilon} dx} \leq B, \tag{26}$$

for all bounded intervals I in $\mathbb{R}$. Moreover the constant G_Ψ is defined by

$$G_\Psi(v) = \sup_{\varepsilon>0} \sup_I \Phi \left[\frac{\Psi^{-1}\left(\fint_I \Psi\left(\frac{v}{\varepsilon}\right) dx \right)}{\fint_I \frac{v}{\varepsilon} dx} \right], \tag{27}$$

where Φ is the complementary Young function of Ψ.

In [13], Kerman and Torchinsky proved the following result about the connection between A_Φ and A_p classes, (see also [14], page 33, Theorem 2.1.1).

Theorem 3.1. *([13], [14]) Let w be a weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition. The following conditions are equivalent:*

i) $w(x) \in A_\Phi$

ii) $w(x) \in A_p$, *where* $p = \underline{\alpha}(\Phi)$.

In [19], (see also [6]) Migliaccio proved the following

Theorem 3.2. *([19], [6]) Let w be a weight and let Φ, Ψ be complementary Young functions verifying Δ_2-condition. We have*

$$w \in G_\Psi \Longrightarrow w \in G_q, \; \forall q < \underline{\alpha}(\Psi). \tag{28}$$

We are going now to prove a result (see Corollary 3.4) which gives a connection between G_p and G_Ψ classes. We begin with the following

Lemma 3.3. *Let $s \geq p \geq 1$, w be a weight, and let Ψ be a Young function such that*

$$\Psi(\lambda t) \leq C \max\{\lambda^p, \lambda^s\} \Psi(t), \; \lambda, t \geq 0. \tag{29}$$

Then the following inequality holds for all bounded intervals $I \subset \mathbb{R}$:

$$\frac{\fint_I \Psi(w(x)) dx}{\Psi\left(\fint_I w(x) dx\right)} \leq \frac{C}{|I|} \left[\int_E \frac{w^s(x)}{\left(\fint_I w(x) dx\right)^s} dx + \int_F \frac{w^p(x)}{\left(\fint_I w(x) dx\right)^p} dx \right] \tag{30}$$

where $E = I \cap \left\{ w(x) > \fint_I w(x) dx \right\}$ and $F = I \cap \left\{ w(x) \leq \fint_I w(x) dx \right\}$.

Proof. The following average integral

$$\fint_I \Psi(w(x))\,dx = \fint_I \Psi\left(w(x)\,\frac{\fint_I w(x)dx}{\fint_I w(x)dx}\right) dx$$

is, by (29) applied with $\lambda = \frac{w(x)}{\fint_I w(x)dx}$ and $t = \fint_I w(x)dx$, smaller than

$$C\fint_I \max\left\{ \left[\frac{w(x)}{\fint_I w(x)dx}\right]^p, \left[\frac{w(x)}{\fint_I w(x)dx}\right]^s \right\} dx\, \Psi\left(\fint_I w(x)dx\right) =$$

$$= \frac{C}{|I|}\left[\int_E \frac{w^s(x)}{\left(\fint_I w(x)dx\right)^s} dx\, \Psi\left(\fint_I w(x)dx\right) + \right.$$

$$\left. + \int_F \frac{w^p(x)}{\left(\fint_I w(x)dx\right)^p} dx\, \Psi\left(\fint_I w(x)dx\right)\right]$$

from which the assertion follows. □

An immediate consequence of Lemma 3.3 is the following

Corollary 3.4. *Let $s \geq p \geq 1$, $w \in G_s$ be a weight, and let Ψ be a Young function such that*

$$\Psi(\lambda t) \leq C\max\{\lambda^p, \lambda^s\}\,\Psi(t),\ \lambda, t \geq 0. \tag{31}$$

Then $w \in G_\Psi$ and

$$\frac{\fint_I \Psi(w(x))dx}{\Psi\left(\fint_I w(x)dx\right)} \leq C\{[G_s(w(x))]^{s-1} + 1\}. \tag{32}$$

Note that hypothesis (31) is satisfied, because of Proposition 2.4, by all functions with growth between the powers t^p and t^s.

Remark 3.5. We observe that if $\Psi(t) = t^r$ in the Corollary 3.4 is a power function, then assumption (31) reads as $p \leq r \leq s$ and inequality (32) is a direct consequence of the Hölder's inequality.

Remark 3.6. Note that Theorems 2.5, 3.1, 3.2, Lemma 3.3 and Corollary 3.4 continue to hold also in $\mathbb{R}^n$.

4. On increasing homeomorphisms

We begin to state some auxiliary results.

Let us recall first the following

Theorem 4.1. *[11] Let $h:\mathbb{R}\longrightarrow\mathbb{R}$ be an increasing homeomorphism onto such that h,h^{-1} are locally absolutely continuous. Then*

$$h'\in A_p \Longleftrightarrow (h^{-1})'\in G_{p'},\ \ \frac{1}{p}+\frac{1}{p'}=1 \tag{33}$$

and

$$A_p(h')=G_{p'}(h^{-1})'. \tag{34}$$

An immediately consequence of Theorem 4.1 is the following

Lemma 4.2. *Let $h:\mathbb{R}\longrightarrow\mathbb{R}$ be an increasing homeomorphism onto such that h,h^{-1} are locally absolutely continuous. Then*

$$h'\in A_\infty \Longleftrightarrow (h^{-1})'\in A_\infty.$$

Note that (34) continues to hold also in the limit case via limiting formulas contained in [17] and [24]. A direct proof of this result is contained in [3] where the following is proved

Theorem 4.3. *[3] Let $h:\mathbb{R}\longrightarrow\mathbb{R}$ be an increasing homeomorphism onto such that h,h^{-1} are locally absolutely continuous. Then*

$$A_\infty(h')=G_1((h^{-1})'). \tag{35}$$

Now we report a result about the improvement of the integrability exponent of a function that is in G_q. The following Theorem is contained in [4].

Theorem 4.4. *[4] Let $v:\mathbb{R}\longrightarrow\mathbb{R}$ be a nonincreasing and nonnegative function. If*

$$\left(\fint_I v^p dx\right)^{\frac{1}{p}}\le B\fint_I v dx,$$

for all bounded intervals $I\subset\mathbb{R}$, there exists $\beta=\beta(p,B)>p$ such that $v\in G_q$, $\forall q\in[p,\beta)$. Moreover the best constant β is the solution of the equation

$$1-B^p\,\frac{x-p}{x}\left(\frac{x}{x-1}\right)^p=0.$$

In the following Corollary we prove that, if the fundamental indices of Ψ are very close, we can pass from G_Ψ to $G_{\overline{\alpha}(\Psi)}$ (compare with Theorem 3.2). Note that in some sense next result is dual to Corollary 3.4, where we passed from G_s to G_Ψ.

Corollary 4.5. *Let w be a weight and let Ψ be a Young function verifying the Δ_2-condition. Let $q>1$ and $\beta=\beta(q,B)>q$ be from Theorem 4.4 such that*

$$q<\underline{\alpha}(\Psi)\leq\overline{\alpha}(\Psi)<\beta, \tag{36}$$

then

$$w\in G_\Psi\Longrightarrow w\in G_{\overline{\alpha}(\Psi)}.$$

Proof. If $w\in G_\Psi$, then by Theorem 3.2, we get $w\in G_{\underline{\alpha}(\Psi)}$. By Hölder's inequality we have $w\in G_q$ and finally by Theorem 4.4 we get the result. □

Proposition 4.6. *Let Ψ be a Young function such that* (36) *holds for some q,β from Theorem 4.4, and*

$$t\Psi'(t)\leq\overline{\alpha}(\Psi)\Psi(t).$$

Then the classes G_Ψ and $G_{\overline{\alpha}(\Psi)}$ coincide.

Proof. By Corollary 4.5 it is $G_\Psi\subset G_{\overline{\alpha}(\Psi)}$. On the other hand, if $w\in G_{\overline{\alpha}(\Psi)}$, setting $p=1$ and $q=\overline{\alpha}(\Psi)$ in Proposition 2.4 we get inequality (31) with $p=1$ and $q=\overline{\alpha}(\Psi)$, from which, by Corollary 3.4, $w\in G_\Psi$. □

Lemma 4.7. *If $\varphi\in C^1([0,\infty))$, $\varphi(0)=0$, is increasing and such that*

$$\exists\ \varepsilon\in]0,1[\ :\ \ \varphi^\varepsilon \text{ is concave}, \tag{37}$$

then the function Γ_ρ defined by

$$\Gamma_\rho: t\in[0,+\infty[\to\Gamma_\rho(t)=\varphi^{-1}(t^\rho)$$

is convex for all $\rho\geq 1/\varepsilon$.

Proof. Let us fix $\rho\geq 1/\varepsilon$. It is

$$\Gamma'_\rho(t)=\frac{d}{dt}(\varphi^{-1}(t^\rho))=(\varphi^{-1})'(t^\rho)\cdot\rho t^{\rho-1}=\frac{1}{\varphi'(\varphi^{-1}(t^\rho))}\cdot\rho t^{\rho-1}\quad\forall t>0 \tag{38}$$

so that, setting $t=\varphi(s)^{1/\rho}$, it is

$$\Gamma'_\rho(\varphi(s)^{1/\rho})=\frac{1}{\varphi'(s)}\cdot\rho[\varphi(s)^{1/\rho}]^{\rho-1}=\frac{\rho}{\varphi'(s)}\varphi(s)^{1/\rho'}\qquad\forall s>0. \tag{39}$$

Since φ is strictly increasing, the assertion is proven if we show that the function on the right hand side of (38) is increasing, or, equivalently, that the right hand side of (39) is increasing. Let us first observe that, in general, if $\varphi \in C^1$ is increasing and concave, then φ^α, where $0 < \alpha \leq 1$, is also concave. This is clear for $\alpha = 1$. If $0 < \alpha < 1$, observe that the derivative $(\varphi^\alpha)' = \alpha\varphi^{\alpha-1}\varphi'$ is decreasing since $\varphi^{\alpha-1}$ is decreasing, $\varphi' > 0$ (because φ is increasing) and φ' is decreasing. As a consequence, since $0 < \frac{1}{\rho} \leq \varepsilon$, the function $\varphi^{\frac{1}{\rho}}$ is concave and therefore its derivative

$$(\varphi^{1/\rho})' = \frac{1}{\rho}\varphi^{\frac{1}{\rho}-1}\varphi' = \frac{1}{\rho}\varphi^{-\frac{1}{\rho'}}\varphi'$$

is decreasing. This implies that its reciprocal is increasing, i.e. the right hand side of (39) is increasing. The lemma is therefore proven. □

We remark that a Young function satisfying the $\triangle_2$-condition along with its complementary function, has not necessarily the derivative satisfying the condition (37). In fact it is possible to consider the following

Example 4.8. It is sufficient to consider the function

$$\varphi(t) = \begin{cases} t^2 & t \in [0,1] \\ e^{2(t-1)} & t \in [1,2] \\ \frac{e^2}{16}t^4 & t \in [2,+\infty[\end{cases}$$

It is straighforward to check that $2\varphi(t) \leq t\varphi'(t) \leq 4\varphi(t)\ \forall t > 0$. Moreover, (37) does not hold because

$$(\varphi^\varepsilon(t))' = 2\varepsilon e^{2\varepsilon(t-1)} > 0 \qquad \forall t \in [1,2].$$

Lemma 4.9. *If $\varphi \in C^1([0,\infty))$, $\varphi(0) = 0$, is increasing and such that*

$$\exists\ \varepsilon \in]0,1[\ :\ \varphi^\varepsilon \textit{ is concave}. \tag{40}$$

For every $\rho \geq \frac{1}{\varepsilon}$ we have

$$\left[⨍_I f^{\frac{1}{\rho}} ds \right]^\rho \leq \varphi\left(⨍_I \varphi^{-1}(f) ds \right), \quad \forall f : \varphi^{-1}(f) \in L^1(\mathbb{R}) \tag{41}$$

for all bounded intervals $I \subset \mathbb{R}$.

Proof. If we set in (41) $f^{\frac{1}{\rho}} = g$, then the thesis of Lemma becomes:

$$\left[⨍_I g ds \right]^\rho \leq \varphi \left(⨍_I \varphi^{-1}(g^\rho) ds \right)$$

equivalently

$$⨍_I g ds \leq \varphi^{\frac{1}{\rho}} \left(⨍_I \varphi^{-1}(g^\rho) ds \right).$$

If we set in the previous inequality $\varphi^{-1}(t^\rho) = \lambda(t)$, then we have to show that

$$⨍_I g\, ds \leq \lambda^{-1} \left(⨍_I \lambda(g) ds \right)$$

and this is true when λ is convex, by Jensen inequality. The fact that λ is convex comes from Lemma 4.7, with $\lambda = \Gamma_\rho$. $\square$

Now we are able to prove a first result, which represents a connection between A_Φ and G_Ψ classes. In the case of power functions, this result generalizes one implication of Theorem 33.

Theorem 4.10. *Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing homeomorphism onto such that h, h^{-1} are locally absolutely continuous. Let Φ, Ψ be complementary Young functions verifying the Δ_2-condition,and let q, β as in Corollary 4.5, such that $q < \underline{\alpha}(\Psi) \leq \overline{\alpha}(\Psi) < \beta$, then*

$$(h^{-1})' \in G_\Psi \Longrightarrow h' \in A_\Phi.$$

Proof. Fix h as in the statement, so that $(h^{-1})' \in G_\Psi$. From Corollary 4.5 we know that $(h^{-1})' \in G_q$, where $q = \overline{\alpha}(\Psi)$. By Theorem 33 we know that $(h^{-1})' \in G_q \Longleftrightarrow h' \in A_p$, with $p = q'$ which in turn is, by (15), equal to $\underline{\alpha}(\Phi)$. Finally, by Theorem 3.1, we have $h' \in A_\Phi$, i.e. the assertion. $\square$

Now let us set

$$\varphi_{p,\alpha}(s) = \frac{s^p}{\log^\alpha(e+s)},\ \alpha > 0,\ p > 1$$

$$\Phi_{p,\alpha}(t) = \int_0^t \varphi_{p,\alpha}(s)\, ds$$

$$\Psi_{p,\alpha}(t) = \text{complementary function of } \Phi_{p,\alpha}(t).$$

Observe that, setting $\varepsilon = \frac{1}{p}$, $\varphi_{p,\alpha}$ satisfies the condition (37) of Lemma 4.7.

The class of Young functions $\{\Phi_{p,\alpha}\}$ plays a key role in the following result, which represents the main Theorem of the Section, and is a kind of counterpart of Theorem 4.10.

Theorem 4.11. *Let $h:\mathbb{R}\longrightarrow\mathbb{R}$ be an increasing homeomorphism onto such that h,h^{-1} are locally absolutely continuous, then*

$$\forall\, p,M>1\ \exists\,\alpha\geq 0: h'\in A_{\Phi_{p,\alpha}},\ A_{\Phi_{p,\alpha}}(h')\leq M\Longrightarrow (h^{-1})'\in G_{\Psi_{p,\alpha}}.$$

Proof. We know, by definition, that

$$A_{\Phi_{p,\alpha}}(h')\leq M\Longleftrightarrow\forall\,\varepsilon>0,\left(\fint_I\varepsilon h'ds\right)\varphi_{p,\alpha}\left(\fint_I\varphi_{p,\alpha}^{-1}\left(\frac{1}{\varepsilon h'}\right)ds\right)\leq M,\forall I\subset\mathbb{R}. \tag{42}$$

By Lemma 4.9 applied to $f=\frac{1}{h'}$, $\rho=p$ (note that $\varepsilon=\frac{1}{p}$ is such that $\varphi_{p,\alpha}^{\varepsilon}$ is concave) we have

$$\left[\fint_I\left(\frac{1}{h'}\right)^{\frac{1}{p}}ds\right]^p\leq\varphi_{p,\alpha}\left(\fint_I\varphi_{p,\alpha}^{-1}\left(\frac{1}{h'}\right)ds\right),\ \forall I\subset\mathbb{R}.$$

which together with the choice $\varepsilon=1$ in (42), gives

$$\left(\fint_I h'ds\right)\left[\fint_I\left(\frac{1}{h'}\right)^{\frac{1}{p}}ds\right]^p\leq M$$

i.e.

$$h'\in A_{p+1},\qquad A_{p+1}(h')\leq M.$$

By Theorem 33 we have

$$h'\in A_{p+1}\Longleftrightarrow(h^{-1})'\in G_{(p+1)'}$$

and

$$A_{p+1}(h')=G_{(p+1)'}((h^{-1})')\leq M.$$

Therefore, by Theorem 4.4,

$$\exists\,\beta=\beta(p,M):(h^{-1})'\in G_q,\ \forall q\in[(p+1)',(p+1)'+\beta[. \tag{43}$$

Fix $\sigma\in]0,\beta[$ and define $\tau=\tau(p,M)>0$ by

$$(p+1)'+\sigma=(p+1-\tau)'$$

and choose $\alpha>0$ sufficiently small, so that

$$(p-\tau)\varphi_{p,\alpha}(t)\leq t\varphi_{p,\alpha}'(t)\quad\forall t\geq 0. \tag{44}$$

The existence of α such that (44) holds, comes from the following

$$\inf_t\frac{t\varphi_{p,\alpha}'(t)}{\varphi_{p,\alpha}(t)}=p-\alpha\sup_t\frac{t}{(e+t)\log(e+t)}\xrightarrow[\alpha\to 0]{}p.$$

Now by Proposition 2.3 from (44) we get

$$(p+1-\tau)\,\Phi_{p,\alpha}(t) \leq t\,\Phi'_{p,\alpha}(t)\ \ \forall t \geq 0$$

$$t\,\Psi'_{p,\alpha}(t) \leq (p+1-\tau)'\,\Psi_{p,\alpha}(t) = [(p+1)'+\sigma]\,\Psi_{p,\alpha}(t)\ \ \forall t \geq 0.$$

Hence, using the right wing inequalities in Theorem 2.5, applied to $w = \frac{(h^{-1})'}{\varepsilon}$, and setting $q = (p+1)'+\sigma$, we get for all $\varepsilon > 0$ and for all $I \subset \mathbb{R}$:

$$\Psi_{p,\alpha}^{-1}\left(\fint_I \Psi_{p,\alpha}\left(\frac{(h^{-1})'}{\varepsilon}\right) ds\right) \leq \left(\fint_I \left(\frac{(h^{-1})'}{\varepsilon}\right)^{(p+1)'+\sigma} ds\right)^{\frac{1}{(p+1)'+\sigma}}. \tag{45}$$

Moreover, by (43), we get that for some $\overline{M} > 0$ it is

$$\left\{\fint_I [(h^{-1})']^{(p+1)'+\sigma} ds\right\}^{\frac{1}{(p+1)'+\sigma}} \leq \overline{M} \fint_I (h^{-1})' ds$$

i.e. for all $\varepsilon > 0$

$$\left\{\fint_I \left[\frac{(h^{-1})'}{\varepsilon}\right]^{(p+1)'+\sigma} ds\right\}^{\frac{1}{(p+1)'+\sigma}} \leq \overline{M} \fint_I \frac{(h^{-1})'}{\varepsilon} ds.$$

Finally, by (45),

$$\Psi_{p,\alpha}^{-1}\left(\fint_I \Psi_{p,\alpha}\left(\frac{(h^{-1})'}{\varepsilon}\right) ds\right) \leq \overline{M} \fint_I \frac{(h^{-1})'}{\varepsilon} ds$$

i.e.

$$(h^{-1})' \in G_{\Psi_{p,\alpha}}.$$

$\square$

The previous proof shows that the following result is true

Theorem 4.12. $\forall\, p, M > 1\ \exists\, \overline{\alpha} > 0 :\ h' \in A_{\Phi_{p,\alpha}},\ A_{\Phi_{p,\alpha}}(h') \leq M \Longrightarrow (h^{-1})' \in G_{\Psi_{p,\alpha}},\ \forall\, \alpha \in [0, \overline{\alpha}]$.

Remark 4.13. Choosing $\alpha = 0$ in Theorem 4.12 we can see that our result generalizes Theorem 4.1 ([11]).

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ROSSELLA CORPORENTE
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Universitá degli Studi di Napoli "Federico II"
Complesso Universitario Monte S. Angelo
Via Cintia - 80126 Napoli - Italy
e-mail: rossella.corporente@dma.unina.it